ラプラシアンの 3 次元極座標表示

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微分演算子ラプラシアンを 3 次元極座標で表すとどうなるかを考える。 3 次元直交座標系ではラプラシアン ∇^2 は

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1}$$

と表される。 3 次元直交座標系 (x, y, z) と 3 次元極座標系 (r, θ, ϕ) の関係は

$$x = r \sin\theta \cos\phi \tag{2}$$
$$y = r \sin\theta \sin\phi \tag{3}$$
$$z = r \cos\theta \tag{4}$$

である。これを (r, θ, ϕ) について解くと

$$r^2 = x^2 + y^2 + z^2 \tag{5}$$
$$\tan^2\theta = \frac{x^2 + y^2}{z^2} \tag{6}$$
$$\tan\phi = \frac{y}{x} \tag{7}$$

となる。 (x, y, z) による偏微分を (r, θ, ϕ) による偏微分で表すと次のようになる。

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \tag{8}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \tag{9}$$
$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \tag{10}$$

つまり $\partial r/\partial x, \partial\theta/\partial x, \partial\phi/\partial x, \partial r/\partial y, \partial\theta/\partial y, \partial\phi/\partial y, \partial r/\partial z, \partial\theta/\partial z, \partial\phi/\partial z$ が求まればよいことになる。

式 (5) を x で偏微分すると

$$\begin{aligned} \frac{\partial}{\partial x}\left(r^2\right) &= \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) \\ 2r\frac{\partial r}{\partial x} &= 2x \\ \frac{\partial r}{\partial x} &= \frac{x}{r} \\ \frac{\partial r}{\partial x} &= \sin\theta\cos\phi \end{aligned} \tag{11}$$

が得られる。同様に y と z で偏微分すると

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(r^2\right) &= \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) \\
2r\frac{\partial r}{\partial y} &= 2y \\
\frac{\partial r}{\partial y} &= \frac{y}{r} \\
\frac{\partial r}{\partial y} &= \sin\theta\sin\phi \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z}\left(r^2\right) &= \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right) \\
2r\frac{\partial r}{\partial z} &= 2z \\
\frac{\partial r}{\partial z} &= \frac{z}{r} \\
\frac{\partial r}{\partial z} &= \cos\theta \qquad (13)
\end{aligned}
$$

が得られる。今度は式 (6) を x で偏微分すると

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\tan^2\theta\right) &= \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{z^2}\right) \\
\frac{\partial\left(\tan^2\theta\right)}{\partial\tan\theta}\frac{\partial\left(\tan\theta\right)}{\partial\theta}\frac{\partial\theta}{\partial x} &= \frac{2x}{z^2} \\
2\tan\theta\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial x} &= \frac{2x}{z^2} \\
\frac{\tan\theta}{\cos^2\theta}\frac{\partial\theta}{\partial x} &= \frac{r\sin\theta\cos\phi}{r^2\cos^2\theta} \\
\frac{\sin\theta}{\cos^3\theta}\frac{\partial\theta}{\partial x} &= \frac{\sin\theta\cos\phi}{r\cos^2\theta} \\
\frac{1}{\cos\theta}\frac{\partial\theta}{\partial x} &= \frac{\cos\phi}{r} \\
\frac{\partial\theta}{\partial x} &= \frac{\cos\theta\cos\phi}{r} \qquad (14)
\end{aligned}
$$

となる。同様に式 (6) を y で微分すると

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\tan^2\theta\right) &= \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{z^2}\right) \\
2\tan\theta\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial y} &= \frac{2y}{z^2} \\
\frac{\sin\theta}{\cos^3\theta}\frac{\partial\theta}{\partial y} &= \frac{r\sin\theta\sin\phi}{r^2\cos^2\theta} \\
\frac{1}{\cos\theta}\frac{\partial\theta}{\partial y} &= \frac{\sin\phi}{r} \\
\frac{\partial\theta}{\partial y} &= \frac{\cos\theta\sin\phi}{r} \qquad (15)
\end{aligned}
$$

が得られる。 z で微分すると

$$
\frac{\partial}{\partial z}\left(\tan^2\theta\right) = \frac{\partial}{\partial z}\left(\frac{x^2+y^2}{z^2}\right)
$$

$$
\begin{aligned}
2\tan\theta\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial z} &= -2\frac{x^2+y^2}{z^3} \\
\frac{\sin\theta}{\cos^3\theta}\frac{\partial\theta}{\partial z} &= -\frac{r^2\sin^2\theta}{r^3\cos^3\theta} \\
\frac{\partial\theta}{\partial z} &= -\frac{\sin\theta}{r}
\end{aligned}
\tag{16}
$$

となる。式 (7) を x で偏微分すると

$$
\begin{aligned}
\frac{\partial}{\partial x}(\tan\phi) &= \frac{\partial}{\partial x}\left(\frac{y}{x}\right) \\
\frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial x} &= -\frac{y}{x^2} \\
\frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial x} &= -\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta\cos^2\phi} \\
\frac{\partial\phi}{\partial x} &= -\frac{\sin\phi}{r\sin\theta}
\end{aligned}
\tag{17}
$$

である。式 (7) を y で偏微分すると

$$
\begin{aligned}
\frac{\partial}{\partial y}(\tan\phi) &= \frac{\partial}{\partial y}\left(\frac{y}{x}\right) \\
\frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial y} &= \frac{1}{x} \\
\frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial y} &= \frac{1}{r\sin\theta\cos\phi} \\
\frac{\partial\phi}{\partial y} &= \frac{\cos\theta}{r\sin\theta}
\end{aligned}
\tag{18}
$$

となる。式 (7) を z で偏微分すると

$$
\begin{aligned}
\frac{\partial}{\partial z}(\tan\phi) &= \frac{\partial}{\partial z}\left(\frac{y}{x}\right) \\
\frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial z} &= 0 \\
\frac{\partial\phi}{\partial y} &= 0
\end{aligned}
\tag{19}
$$

が得られる。
式 (8), (11), (14), (17) から

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \\
&= \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}
\end{aligned}
\tag{20}
$$

が得られる。式 (9), (12), (15), (18) から

$$
\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} \\
&= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}
\end{aligned}
\tag{21}
$$

が得られる。式 (10), (13), (16), (19) から

$$
\begin{aligned}
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \\
&= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}
\end{aligned} \tag{22}
$$

が得られる。

式 (1) に式 (20), (21), (22) を代入すれば、ラプラシアンを 3 次元極座標で表すことができる。そのために $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, $\partial^2/\partial z^2$ を計算しておく。

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2} \\
=& \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\
=& \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2} + \sin\theta\cos\theta\cos^2\phi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \sin\phi\cos\phi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \phi}\right) \\
&+\frac{\cos\theta\cos^2\phi}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial r}\right) + \frac{\cos\theta\cos^2\phi}{r^2}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial}{\partial \theta}\right) - \frac{\cos\theta\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right) \\
&-\frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial}{\partial r}\right) - \frac{\cos\theta\sin\phi}{r^2\sin\theta}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial}{\partial \theta}\right) + \frac{\sin\phi}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial \phi}\right)
\end{aligned} \tag{23}
$$