Kepler's Laws

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1 Kepler's Laws of Planetary Motion

1.1 First law

All planets move along elliptical path with the Sun at one focus. (Fig. 1)

Heliocentric distance of a planet $r_\odot$ can be expressed as

$$r_\odot = \frac{a(1-e^2)}{1+e\cos f}. \tag{1}$$

Here, a is semimajor axis, e is eccentricity, and f is true anomaly, respectively.

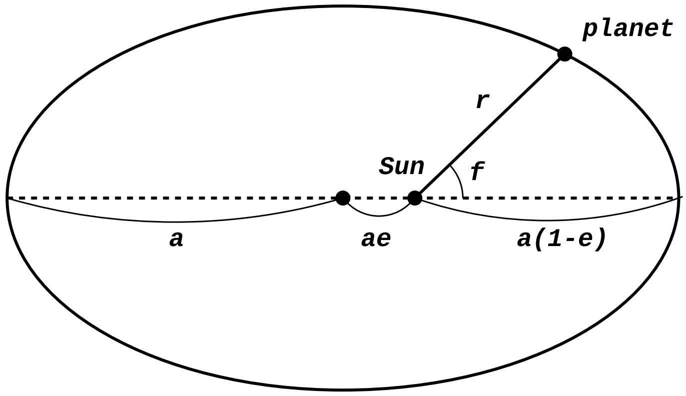

Figure 1: An ellipitical orbit of a planet.

1.2 Second law

A line connecting any given planet and the Sun sweeps out area S at a constant rate.

$$\frac{dS}{dt} = const. \tag{2}$$

1.3 Third law

The square of a planet's orbital period about the Sun P_{yr} in year is equal to the cube of its semimajor axis a_{au} in au.

$$P_{yr}^2 = a_{au}^3 \tag{3}$$

2 2-Dimentional Polar Coordinate

The relationship between 2-dimentional Cartesian coordinate (x, y) and polar coordinate (r, θ) is

$$x = r\cos\theta, \tag{4}$$
$$y = r\sin\theta. \tag{5}$$

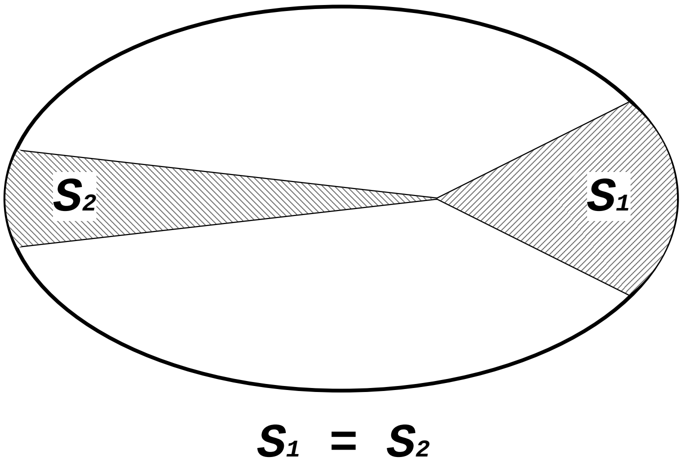

Figure 2: Kepler's second law of planetary motion.

Now, we differntiate x and y with respect to the time t. Then, we obtain followings.

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} \\ &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \qquad (6) \\ \dot{y} &= \frac{dy}{dt} \\ &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \qquad (7) \end{aligned}$$

We differentiate $\dot{x}$ and $\dot{y}$.

$$\begin{aligned} \ddot{x} &= \frac{d}{dt}(\dot{x}) \\ &= \frac{d}{dt}(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) \\ &= \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\frac{d}{dt}(\dot{\theta}\sin\theta) \\ &= \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ &= \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ &= (\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta \qquad (8) \end{aligned}$$

$$\begin{aligned} \ddot{y} &= \frac{d}{dt}(\dot{y}) \\ &= \frac{d}{dt}(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) \\ &= \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta + r\frac{d}{dt}(\dot{\theta}\cos\theta) \\ &= \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\ &= \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\ &= (\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos\theta \qquad (9) \end{aligned}$$

We make $\ddot{x}\cos\theta$ and $\ddot{y}\sin\theta$.

$$\begin{aligned} \ddot{x}\cos\theta &= (\ddot{r} - r\dot{\theta}^2)\cos^2\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta\cos\theta \qquad (10) \\ \ddot{y}\sin\theta &= (\ddot{r} - r\dot{\theta}^2)\sin^2\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta\cos\theta \qquad (11) \end{aligned}$$

By adding the equation 11 to the equation 10, we get

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = (\ddot{r} - r\dot{\theta}^2). \qquad (12)$$

Similarly, we make $\ddot{x}\sin\theta$ and $\ddot{y}\cos\theta$.

$$\begin{aligned} \ddot{x}\sin\theta &= (\ddot{r} - r\dot{\theta}^2)\sin\theta\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin^2\theta & (13) \\ \ddot{y}\cos\theta &= (\ddot{r} - r\dot{\theta}^2)\sin\theta\cos\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos^2\theta & (14) \end{aligned}$$

By subtracting the equation 14 from the equation 13, we get

$$\ddot{x}\sin\theta - \ddot{y}\cos\theta = -(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \tag{15}$$

3 Equation of Motion

The equation of motion using the 2-D Cartesian coordinate can be written as

$$\begin{aligned} m\ddot{x} &= F_x, & (16) \\ m\ddot{y} &= F_y. & (17) \end{aligned}$$

Here, m is the mass of a planet, F_x is the force acting on a planet in x direction, and F_y is the force acting on a planet in y direction. We consider to express r and θ components of the force in 2-D polar coordinate. From Fig. 3, F_r and F_θ are expressed as

$$\begin{aligned} F_r &= F_x\cos\theta + F_y\sin\theta, & (18) \\ F_\theta &= -F_x\sin\theta + F_y\cos\theta. & (19) \end{aligned}$$

From the equations 18 and 19, we get

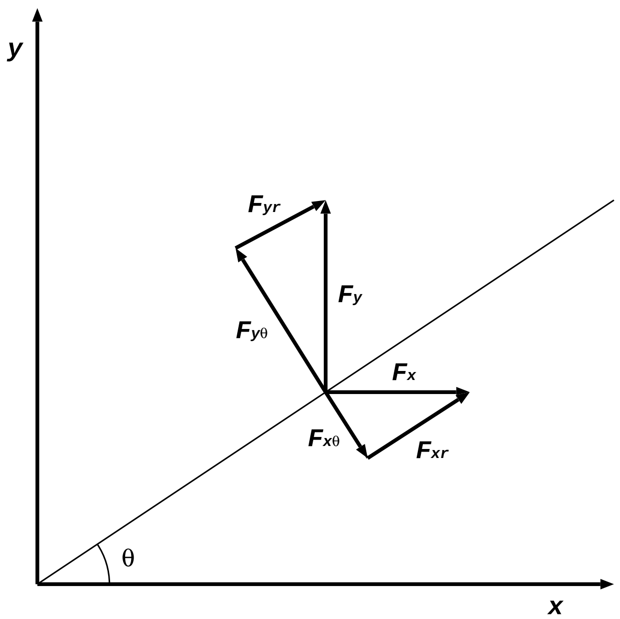

Figure 3: The force in Cartesian and polar coordinates.

$$\begin{aligned} F_r\cos\theta &= F_x\cos^2\theta + F_y\sin\theta\cos\theta & (20) \\ F_\theta\sin\theta &= -F_x\sin^2\theta + F_y\sin\theta\cos\theta. & (21) \end{aligned}$$

Subtracting one from other, and we have

$$F_x = F_r\cos\theta - F_\theta\sin\theta \tag{22}$$

Similarly, we get

$$\begin{aligned} F_r\sin\theta &= F_x\sin\theta\cos\theta + F_y\sin^2\theta & (23) \\ F_\theta\cos\theta &= -F_x\sin\theta\cos\theta + F_y\cos^2\theta. & (24) \end{aligned}$$

Adding these two equation, and we have

$$F_y = F_r \sin\theta + F_\theta \cos\theta \tag{25}$$

The equation of motion is now written as

$$\begin{aligned} m\ddot{x} &= F_r \cos\theta - F_\theta \sin\theta, & (26) \\ m\ddot{y} &= F_r \sin\theta + F_\theta \cos\theta. & (27) \end{aligned}$$

Making $m\ddot{x}\cos\theta$ and $m\ddot{y}\sin\theta$, we obtain

$$\begin{aligned} m\ddot{x}\cos\theta &= F_r \cos^2\theta - F_\theta \sin\theta\cos\theta, & (28) \\ m\ddot{y}\sin\theta &= F_r \sin^2\theta + F_\theta \sin\theta\cos\theta. & (29) \end{aligned}$$

By adding these two equations, we get

$$m(\ddot{x}\cos\theta + \ddot{y}\sin\theta) = F_r. \tag{30}$$

Similarly, making $m\ddot{x}\sin\theta$ and $m\ddot{y}\cos\theta$, we obtain

$$\begin{aligned} m\ddot{x}\sin\theta &= F_r \sin\theta\cos\theta - F_\theta \sin^2\theta, & (31) \\ m\ddot{y}\cos\theta &= F_r \sin\theta\cos\theta + F_\theta \cos^2\theta. & (32) \end{aligned}$$

By subtracting one from the other, we get

$$m(\ddot{x}\sin\theta - \ddot{y}\cos\theta) = -F_\theta. \tag{33}$$

Using the results of previous section, we have

$$m(\ddot{r} - r\dot{\theta}^2) = F_r, \tag{34}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta. \tag{35}$$

Now, we calculate $\frac{d}{dt}\left(r^2\dot{\theta}\right)$.

$$\begin{aligned} \frac{d}{dt}\left(r^2\dot{\theta}\right) &= 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} \\ &= r(r\ddot{\theta} + 2\dot{r}\dot{\theta}) & (36) \end{aligned}$$

Therefore, we get

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{37}$$

The equation 35 is now rewritten as

$$m\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = F_\theta. \tag{38}$$

The gravitational force is a central force, and the θ component of the gravitational force is always zero if the origin of the coordinate is chosen to be the location of the Sun.

$$F_\theta = 0 \tag{39}$$

Then, the equation 38 is now written as

$$m\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0. \tag{40}$$

By integrating this equation, we yeild

$$r^2\dot{\theta} = h. \tag{41}$$

Here, h is a constant. This is Kepler's second law. Note that the area that is swept by a line connecting a planet and the Sun in a short time Δt is $\frac{1}{2}r^2\dot{\theta}$. (Fig. 4)

From the equation 41, we get

$$\begin{aligned} \dot{\theta} &= \frac{h}{r^2}, \\ \dot{\theta}^2 &= \frac{h^2}{r^4}, \\ r^2\dot{\theta}^2 &= \frac{h^2}{r^2}. & (42) \end{aligned}$$

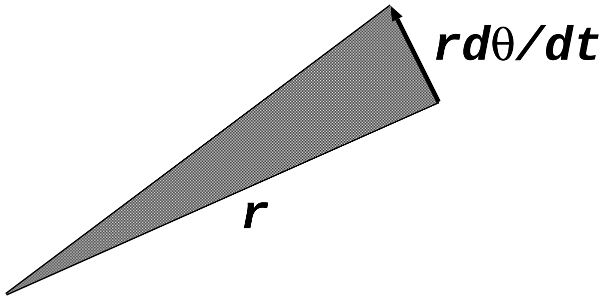

Figure 4: The area that is swept out by a line connecting a planet and the Sun.

The gravitational force acting on the planet located at the distance of r from the Sun is shown as

$$F_r = -\frac{GMm}{r^2}. \tag{43}$$

Here, G is the gravitational constant, M is the mass of the Sun, and m is the mass of a planet, respectively.
From the equation 34, we have

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2}. \tag{44}$$

We devide both sides of the equation by m.

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \tag{45}$$

Then, we multiply both sides of the equation by $\dot{r}$.

$$\dot{r}\ddot{r} - r\dot{r}\dot{\theta}^2 = -\frac{GM}{r^2}\dot{r} \tag{46}$$

Here, we know following relations.

$$\frac{d}{dt}\left(\dot{r}^2\right) = 2\dot{r}\ddot{r} \tag{47}$$

$$\frac{d}{dt}\left(\frac{h^2}{r^2}\right) = -2\frac{h^2}{r^3}\dot{r} = -2\frac{r^4\dot{\theta}^2}{r^3}\dot{r} = -2r\dot{r}\dot{\theta}^2 \tag{48}$$

$$\frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\dot{r} \tag{49}$$

Therefore, the equation 46 can now be written as

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) + \frac{d}{dt}\left(\frac{1}{2}\frac{h^2}{r^2}\right) = \frac{d}{dt}\left(\frac{GM}{r}\right). \tag{50}$$

We move all the terms to the left hand side of the equation.

$$\frac{d}{dt}\left[\frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{h^2}{r^2} - \frac{GM}{r}\right] = 0 \tag{51}$$

By integrating the equation, we obtain

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{h^2}{r^2} - \frac{GM}{r} = E. \tag{52}$$

Here, E is a constant. Now, we have

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{GM}{r} = E. \tag{53}$$

This is actually the equation for the conservation of energy.

4 Use of new variables φ and s

Here, we introduce new variables φ and s.

$$\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\varphi} \tag{54}$$

$$s = \frac{1}{r} \tag{55}$$

$$\frac{ds}{dr} = -\frac{1}{r^2} = -s^2 \tag{56}$$

The relationship between dr/dt and $ds/d\varphi$ can be written as

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{ds}\frac{ds}{dt} \\ &= -r^2\frac{h}{r^2}\frac{ds}{d\varphi} \\ &= -h\frac{ds}{d\varphi}.\end{aligned} \tag{57}$$

Now, we rewrite the equation of energy conservation using φ and s.

$$\frac{1}{2}h^2\left(\frac{ds}{d\varphi}\right)^2 + \frac{1}{2}h^2s^2 - GMs = E \tag{58}$$

$$\left(\frac{ds}{d\varphi}\right)^2 = \frac{2}{h^2}E + \frac{2GM}{h^2}s - s^2 \tag{59}$$

$$\frac{ds}{d\varphi} = \sqrt{\frac{2}{h^2}E + \frac{2GM}{h^2}s - s^2} \tag{60}$$

$$\begin{aligned}d\varphi &= \frac{ds}{\sqrt{\frac{2}{h^2}E + \frac{2GM}{h^2}s - s^2}} \\ &= \frac{ds}{\sqrt{-\left[\left(s - \frac{GM}{h^2}\right)^2 - \frac{G^2M^2}{h^4} - \frac{2E}{h^2}\right]}}\end{aligned} \tag{61}$$

Here, we use a new variable s'.

$$s' = s - \frac{GM}{h^2} \tag{62}$$

We now have

$$d\varphi = \frac{ds'}{\sqrt{\left(\frac{2E}{h^2} + \frac{G^2M^2}{h^4}\right) - s'^2}}. \tag{63}$$

To integrate above equation, we use following formula.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = -\cos^{-1}\left(\frac{x}{a}\right) + \alpha \tag{64}$$

We first prove this formula. For $x = a\cos\theta$, it is

$$\cos\theta = \frac{x}{a}, \tag{65}$$

and, therefore, we have

$$\theta = \cos^{-1}\left(\frac{x}{a}\right). \tag{66}$$

The term $\sqrt{a^2 - x^2}$ can be written as

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \cos^2 \theta} \\
&= a\sqrt{1 - \cos^2 \theta} \\
&= a \sin \theta. \qquad (67)
\end{aligned}$$

We differentiate x with respect to θ.

$$\begin{aligned}
\frac{dx}{d\theta} &= -a \sin \theta \\
dx &= -a \sin \theta \cdot d\theta \qquad (68)
\end{aligned}$$

Now, we get

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{1}{a \sin \theta} \cdot (-a \sin \theta)\, d\theta \\
&= -\int d\theta \\
&= -\theta + C \\
&= -\cos^{-1}\left(\frac{x}{a}\right) + C. \qquad (69)
\end{aligned}$$

We integrate $d\varphi$.

$$\begin{aligned}
\varphi &= \int d\varphi \\
&= \int \frac{ds'}{\sqrt{\left(\frac{2E}{h^2} + \frac{G^2M^2}{h^4}\right) - s'^2}} \\
&= -\cos^{-1}\left(\frac{s'}{\sqrt{\frac{2E}{h^2} + \frac{G^2M^2}{h^4}}}\right) + \omega \\
&= -\cos^{-1}\left(\frac{s - \frac{GM}{h^2}}{\sqrt{\frac{2E}{h^2} + \frac{G^2M^2}{h^4}}}\right) + \omega \\
&= -\cos^{-1}\left(\frac{\frac{s}{GM} - \frac{1}{h^2}}{\sqrt{\frac{2E}{h^2G^2M^2} + \frac{1}{h^4}}}\right) + \omega \\
&= -\cos^{-1}\left(\frac{\frac{h^2s}{GM} - 1}{\sqrt{\frac{2Eh^2}{G^2M^2} + 1}}\right) + \omega
\end{aligned} \qquad (70)$$

We modify the equation as follows.

$$\begin{aligned}
-(\varphi - \omega) &= \cos^{-1}\left(\frac{\frac{h^2s}{GM} - 1}{\sqrt{\frac{2Eh^2}{G^2M^2} + 1}}\right) \\
\frac{\dfrac{h^2s}{GM} - 1}{\sqrt{\dfrac{2Eh^2}{G^2M^2} + 1}} &= \cos(\varphi - \omega) \\
\frac{h^2s}{GM} - 1 &= \sqrt{\frac{2Eh^2}{G^2M^2} + 1} \cdot \cos(\varphi - \omega) \\
\frac{h^2s}{GM} &= 1 + \sqrt{\frac{2Eh^2}{G^2M^2} + 1} \cdot \cos(\varphi - \omega) \\
\frac{h^2}{GM}\frac{1}{r} &= 1 + \sqrt{\frac{2Eh^2}{G^2M^2} + 1} \cdot \cos(\varphi - \omega) \qquad (71)
\end{aligned}$$

We finally get

$$r = \frac{\frac{h^2}{GM}}{1 + \sqrt{1 + \frac{2Eh^2}{G^2M^2}} \cos(\varphi - \omega)}. \tag{72}$$

Note that the equation for an ellipse can be expressed as

$$r = \frac{l}{1 + e\cos\theta} \tag{73}$$

using the polar coordinate (r, θ).

5 Equation of an ellipse

Definition of ellipse: a closed plane curve generated by a point moving in such a way that the sums of its distances from two fixed points (foci or focal points) is a constant. (Fig. 5)

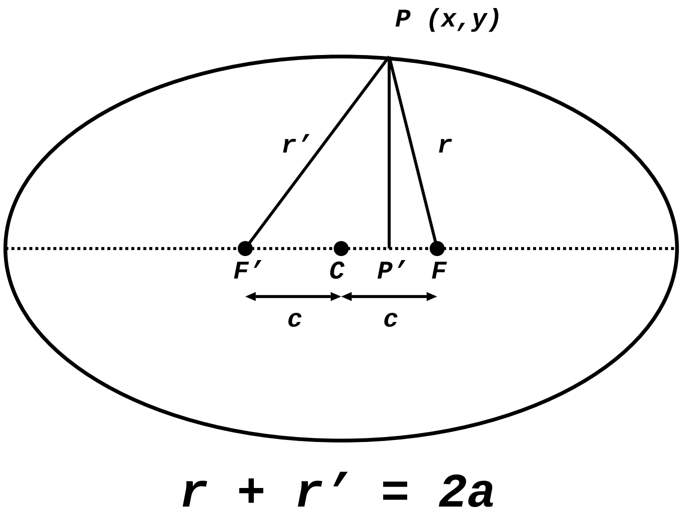

Figure 5: Definition of ellipse.

We start from the definition of ellipse, and show the equations of ellipse. The definition is

$$r + r' = 2a. \tag{74}$$

r and r' are

$$r = \sqrt{(c - x)^2 + y^2}, \tag{75}$$

$$r' = \sqrt{(c + x)^2 + y^2}. \tag{76}$$

From the definition, we get

$$\sqrt{(c - x)^2 + y^2} + \sqrt{(c + x)^2 + y^2} = 2a. \tag{77}$$

We modify this equation.

$$\left(\sqrt{(c - x)^2 + y^2} + \sqrt{(c + x)^2 + y^2}\right)^2 = 4a^2 \tag{78}$$

$$(c - x)^2 + y^2 + (c + x)^2 + y^2 + 2\sqrt{(c - x)^2 + y^2}\sqrt{(c + x)^2 + y^2} = 4a^2 \tag{79}$$

$$2c^2 + 2x^2 + 2y^2 + 2\sqrt{(c - x)^2 + y^2}\sqrt{(c + x)^2 + y^2} = 4a^2 \tag{80}$$

$$c^2 + x^2 + y^2 + \sqrt{(c - x)^2 + y^2}\sqrt{(c + x)^2 + y^2} = 2a^2 \tag{81}$$

$$x^2 + y^2 + c^2 - 2a^2 = -\sqrt{(c - x)^2 + y^2}\sqrt{(c + x)^2 + y^2} \tag{82}$$

$$\left(x^2 + y^2 + c^2 - 2a^2\right)^2 = \left[(c - x)^2 + y^2\right]\left[(c + x)^2 + y^2\right] \tag{83}$$